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NOTES ON OPTIMUM SIGNALS DETECTION THEORY,(U)
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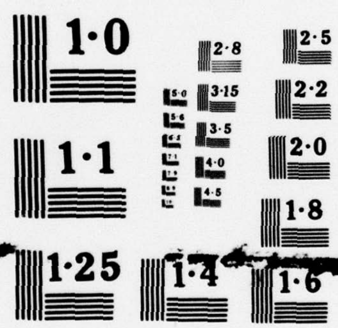
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(6) Notes on Optimum Signal Detection Theory.

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The purpose of this paper is to clarify the relationship between three different criteria for optimization of acoustic signal detection. Specifically, the maximization of array gain , the minimization of signal distortion, and the evaluation of the Neyman-Pearson likelihood ratio are shown to yield equivalent results at a single frequency.

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1. INTRODUCTION

There are several criteria for optimization of a processor for an array of sensors. J.J. Faran and R. Hills, Jr. have used the criterion of maximization of array gain to design real weightings for individual sensors.[1] N. Wiener used the criterion of minimizing signal distortion to design filters.[2] Other authors, notably F. Bryn, have used evaluation of the Neyman-Pearson likelihood ratio to minimize risk.[3] This paper examines the relationship, at a single frequency, between these three criteria. The filters required by each of the three developments are shown to be exactly the same for single frequency considerations.

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[1] J.J. Faran and R. Hills, "Wide-Band Directivity of Receiving Arrays," Harvard Univ. Acoust. Res. Lab. Tech. Mem. 31 (1 May 1953)

[2] N. Wiener, Extrapolation, Interpolation, and Smoothing of Stationary Time Series, (MIT Press & Wiley, March 1949)

[3] F. Bryn, "Optimum Signal Processing of Three-Dimensional Arrays Operating on Gaussian Signals and Noise," J. Acoust. Soc. Am. 34, 289-297 (1962)

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2. FORM OF GAIN EQUATION

Let us assume that we have K hydrophones in a sonar array. These hydrophones have output waveforms $\ell_i(t)$, $i = 1,2,\ldots,K$. Since we may be interested in spectral components, we shall allow these waveforms to be complex. Let Z_i, $i = 1,2,\ldots,K$ denote the (complex) weight functions which the ℓ_i's are to be multiplied by before summing and squaring. The average power output over a time interval between $-T$ and T is

$$W = (1/2T)\int_{-T}^{T}\left\{\sum_{i=1}^{K} Z_i \ell_i(t)\right\}^*\left\{\sum_{j=1}^{K} Z_j \ell_j(t)\right\} dt$$

$$= (1/2T)\int_{-T}^{T}\left\{\sum_{i=1}^{K}\sum_{j=1}^{K} Z_i^* Z_j \ell_i^*(t)\ell_j(t)\right\} dt$$

$$= \sum_{i=1}^{K}\sum_{j=1}^{K} Z_i^* Z_j \,(1/2T)\int_{-T}^{T} \ell_i^*(t)\, \ell_j(t)\, dt \quad .$$

Let $s_i(t)$ denote the signal component of $\ell_i(t)$, and $n_i(t)$ denote the noise component of $\ell_i(t)$. Thus $\ell_i(t) = n_i(t) + s_i(t)$. The output signal-to-noise ratio of the array is defined to be the limit of the ratio of the difference between the average power output when signal is present and when signal is absent to the average power output when signal is absent, i.e.

$$A = \lim_{T\to\infty}\Bigg\{ \Big[\sum_{i=1}^{K}\sum_{j=1}^{K} Z_i^* Z_j \,(1/2T)\int_{-T}^{T} [s_i(t)+n_i(t)]^*[s_j(t)+n_j(t)]\, dt$$

$$- \sum_{i=1}^{K}\sum_{j=1}^{K} Z_i^* Z_j \,(1/2T)\int_{-T}^{T} n_i^*(t)\, n_j(t)\, dt\Big] \Big/$$

$$\Big[\sum_{i=1}^{K}\sum_{j=1}^{K} Z_i^* Z_j \,(1/2T)\int_{-T}^{T} n_i^*(t)\, n_j(t) dt\Big]\Bigg\}$$

$$A = \lim_{T\to\infty} \left\{ \left\{ \sum_{i=1}^{K} \sum_{j=1}^{K} z_i^* z_j \, (1/2T) \int_{-T}^{T} [s_i^*(t) s_j(t) + s_i^*(t) n_j(t) + n_i^*(t) s_j(t)] dt \right\} \Big/ \left\{ \sum_{i=1}^{K} \sum_{j=1}^{K} z_i^* z_j \, (1/2T) \int_{-T}^{T} n_i^*(t) n_j(t) dt \right\} \right\}$$

$$= \left\{ \sum_{i=1}^{K} \sum_{j=1}^{K} z_i^* z_j \left\{ \lim_{T\to\infty} (1/2T) \int_{-T}^{T} s_i^*(t)\, s_j(t) dt + \lim_{T\to\infty} (1/2T) \int_{-T}^{T} s_i^*(t) n_j(t) dt + \lim_{T\to\infty} (1/2T) \int_{-T}^{T} n_i^*(t) s_j(t) dt \right\} \right\} \Big/ \left\{ \sum_{i=1}^{K} \sum_{j=1}^{K} z_i^* z_j \lim_{T\to\infty} (1/2T) \int_{-T}^{T} n_i^*(t) n_j(t) dt \right\}$$

Assumption #1: The signal and noise waveforms are uncorrelated, i.e.,

$$0 = \lim_{T\to\infty} (1/2T) \int_{-T}^{T} s_i^*(t) n_j(t) dt = \lim_{T\to\infty} (1/2T) \int_{-T}^{T} n_i^*(t) s_j(t) dt \quad \text{for all}$$

i and j.

Then if we let

$$\Theta_{ij} = \lim_{T\to\infty} (1/2T) \int_{-T}^{T} s_i^*(t) s_j(t)\, dt$$

and

$$\varphi_{ij} = \lim_{T\to\infty} (1/2T) \int_{-T}^{T} n_i^*(t)\, n_j(t)\, dt,$$

we can write A as

$$A = \left\{ \sum_{i=1}^{K} \sum_{j=1}^{K} z_i^* z_j \Theta_{ij} \right\} \Big/ \left\{ \sum_{i=1}^{K} \sum_{j=1}^{K} z_i^* z_j \varphi_{ij} \right\} .$$

Let S denote the signal power present in a standard hydrophone and N denote the noise power present in the same hydrophone. The array gain is defined to be the signal-to-noise ratio of the array divided by the signal-to-noise ratio of the standard hydrophone, i.e.,

$$G = A/(S/N) = \left\{ \left[\sum_{i=1}^{K} \sum_{j=1}^{K} z_i^* z_j \Theta_{ij} \right] \Big/ \left[\sum_{i=1}^{K} \sum_{j=1}^{K} z_i^* z_j \varphi_{ij} \right] \right\} \cdot (N/S)$$

Let $p_{ij} = \Theta_{ij}/S$ and $q_{ij} = \phi_{ij}/N$. Thus

$$G = \sum_{i=1}^{K} \sum_{j=1}^{K} z_i^* z_j p_{ij} \Big/ \sum_{i=1}^{K} \sum_{j=1}^{K} z_i^* z_j q_{ij} .$$

To simplify this notation further, let

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ \cdots \\ \cdots \\ z_K \end{bmatrix}, \quad P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1K} \\ p_{21} & p_{22} & \cdots & p_{2K} \\ \cdots \\ \cdots \\ p_{K1} & p_{K2} & \cdots & p_{KK} \end{bmatrix}, \text{ and } Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1K} \\ q_{21} & q_{22} & \cdots & q_{2K} \\ \cdots \\ \cdots \\ q_{K1} & q_{K2} & \cdots & q_{KK} \end{bmatrix}.$$

Note that P and Q are correlation matrices and thus are positive definite hermitian. We can now write

$$G = Z^{*T}PZ \Big/ Z^{*T}QZ .$$

Frequently the noise and signal fields are assumed to be homogeneous, i.e., the same noise power and signal power are observed by all hydrophones. Then if the standard hydrophone is an element of the array, this implies that $p_{ii} = q_{ii} = 1$. This assumption is not needed, however, in the present development.

3. MAXIMIZATION OF GAIN

To "optimize" our system, we shall choose the vector Z which, for given P and Q, will maximize G.

$$dG = \{(Z^{*T}QZ)d(Z^{*T}PZ) - (Z^{*T}PZ)d(Z^{*T}QZ)\} / (Z^{*T}QZ)^2$$

$$= \{(Z^{*T}QZ)[Z^{*T}PdZ + (dZ)^{*T}PZ] - (Z^{*T}PZ)[Z^{*T}QdZ + (dZ)^{*T}QZ]\} / (Z^{*T}QZ)^2$$

G will have an extremal value only where dG = 0 for all choices of dZ, hence the numerator must be zero for all choices of dZ.

$$\{(Z^{*T}QZ)Z^{*T}P - (Z^{*T}PZ)Z^{*T}Q\} dZ + (dZ)^{*T}\{(Z^{*T}QZ)PZ - (Z^{*T}PZ)QZ\} = 0.$$

$$(dZ)^T\{(Z^{*T}QZ)P^{*T}Z - (Z^{*T}PZ)Q^{*T}Z\}^* + (dZ)^{*T}\{(Z^{*T}QZ)PZ - (Z^{*T}PZ)QZ\} = 0.$$

Since P and Q are hermitian

$$(dZ)^T\{(Z^{*T}QZ)PZ - (Z^{*T}PZ)QZ\}^* = (dZ)^{*T}\{(Z^{*T}QZ)PZ - (Z^{*T}PZ)QZ\}.$$

The only way this can be true for all possible choices of dZ is for the quantity in braces to be the zero vector, i.e.,

$$(Z^{*T}QZ)PZ = (Z^{*T}PZ)QZ ,$$

or $$PZ = \{(Z^{*T}PZ) / (Z^{*T}QZ)\} QZ .$$

Assumption #2: The matrix Q is nonsingular.

$$Q^{-1}PZ = \{(Z^{*T}PZ) / (Z^{*T}QZ)\} Z .$$

Let $G_o = (Z^{*T}PZ)/(Z^{*T}QZ)$. Then $Q^{-1}P\,Z = G_oZ$, i.e., the optimum gain, G_o, is the largest eigenvalue of the matrix $Q^{-1}P$ and the set of filter weights which produce this gain form the corresponding eigenvector.

Clearly, for large K, a precise investigation of the general nature of G_o is not possible. However, by proper choice of P we can simplify the problem considerably.

Assumption #3: The signal field is produced by a monochromatic wavefront moving across the hydrophone array.

Then $s_i(t) = S^{\frac{1}{2}} e^{j\omega(t-\tau_i)}$, where τ_i is the time for the wavefront to reach the i^{th} hydrophone from some arbitrary point in space.

So

$$P_{ij} = \lim_{T\to\infty} (S_i^{\frac{1}{2}}S_j^{\frac{1}{2}}/2TS)\int_{-T}^{T} e^{-j\omega(t-\tau_i)}e^{j\omega(t-\tau_j)}\,dt$$

$$= \lim_{T\to\infty} (S_i^{\frac{1}{2}}S_j^{\frac{1}{2}}/2TS)\int_{-T}^{T} e^{j\omega(\tau_i-\tau_j)}\,dt$$

$$= (S_i^{\frac{1}{2}}S_j^{\frac{1}{2}}/S)\, e^{j\omega(\tau_i-\tau_j)} \lim_{T\to\infty}(1/2T)\int_{-T}^{T} dt$$

$$= (S_i^{\frac{1}{2}}S_j^{\frac{1}{2}}/S)\, e^{j\omega(\tau_i-\tau_j)} .$$

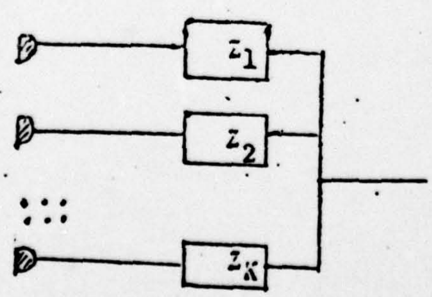
Let $c_i = (S_i^{\frac{1}{2}}/S^{\frac{1}{2}})$. Define $V = \begin{bmatrix} c_1e^{j\omega\tau_1} \\ c_2e^{j\omega\tau_2} \\ \vdots \\ \vdots \\ c_Ke^{j\omega\tau_K} \end{bmatrix}$. Then $P = V\,V^{*T}$.

This implies that P is of rank 1, and hence $Q^{-1}P$ is of rank 1. Thus $Q^{-1}P$ has only one nonzero eigenvalue which must be G_o. Further, we can see that if $R = Q^{-1}$, then $Z = RV$ and $G_o = V^{*T}RV$, for

$$\begin{aligned} Q^{-1}PZ &= G_oZ \\ (RP)(RV) &\overset{?}{=} (V^{*T}RV)(RV) \\ QR\, VV^{*T}\, RV &\overset{?}{=} (V^{*T}RV)\, QRV \\ VV^{*T}RV &\overset{?}{=} (V^{*T}RV)\, V \\ VV^{*T}RV &= VV^{*T}RV \, . \end{aligned}$$

In this way we can arrive at the transfer functions of a set of filters which will give the greatest possible array gain at each frequency under the four assumptions given above. This will be done by repeating the process for each frequency of concern, to get a vector $Z(f)$, where $Z(f) = \begin{bmatrix} Z_1(f) \\ Z_2(f) \\ \cdots \\ \cdots \\ Z_K(f) \end{bmatrix}$.

Then the desired system is



This analysis, however, says nothing about the relative weighting of the frequencies, since any multiple of these filters will produce the same signal-to-noise ratio in the output. Thus, a frequency weighting filter is also desired at the output.

F. Bryn indicates that a desirable filter is the Eckart filter.[4]

The transfer function of this filter is $\mathcal{S}^{\frac{1}{2}}(f)/\mathcal{N}(f)$, where $\mathcal{S}(f)$ and $\mathcal{N}(f)$ are the signal and noise power, respectively, at the summed output of the Z^is .

Note that

$$\begin{aligned}
\mathcal{S}(f) &= S(f)\ Z^{*T}(f)\ P(f)\ Z(f) \\
&= S(f)\{R(f)\ V(f)\}^{*T}\ P(f)\{R(f)V(f)\} \\
&= S(f)\ V^{*T}(f)\ R(f)\ V(f)\ V^{*T}(f)\ R(f)\ V(f) \\
&= S(f)\ G_o^{\ 2}(f),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{N}(f) &= N(f)\ Z^{*T}(f)\ Q(f)\ Z(f) \\
&= N(f)\{R(f)V(f)\}^{*T}\ Q(f)\{R(f)V(f)\} \\
&= N(f)\ V^{*T}(f)\ R(f)\ Q(f)\ R(f)\ V(f) \\
&= N(f)\ V^{*T}(f)\ R(f)\ V(f) \\
&= N(f)\ G_o(f)\ .
\end{aligned}$$

Thus $(\mathcal{S}(f)/\mathcal{N}^2(f)) = (G_o^{\ 2}(f)\ S(f)/\ G_o^{\ 2}(f)\ N^2(f)) = S(f)/N^2(f)$.

However, as will be seen later, it appears that a better system for detection may be $\mathcal{S}^{\frac{1}{2}}(f)/\{\mathcal{N}^{\frac{1}{2}}(f)(\mathcal{S}(f)+\mathcal{N}(f))^{\frac{1}{2}}\}$. Note that this could also be written as

$$\mathcal{S}^{\frac{1}{2}}(f)/\{\mathcal{N}(f)\ (1+[\mathcal{S}(f)/\mathcal{N}(f)])^{\frac{1}{2}}\} = S^{\frac{1}{2}}(f)/\{N(f)\ (1+[S(f)/N(f)]\,G_o(f))^{\frac{1}{2}}\}.$$

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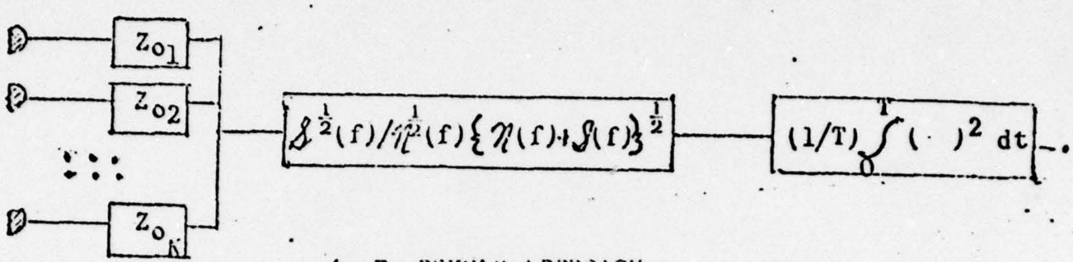
[4]C. Eckart, SIO Ref. 52-11, University of California, Marine Physical Laboratory, Scripps Institution of Oceanography(1952).

The system could then be represented as



4. F. BRYN'S APPROACH

F. Bryn views the basic problem as deciding the answer to the question, "Were these ℓ_i's produced by random variation in a noise field, or by the sum of a signal and random variations in the noise field?"

To answer this question, Bryn considers each frequency separately. We shall use the notation in the previous sections to abridge Bryn's derivation and avoid a small signal assumption. Let us assume that the ℓ_i's are observed over a time interval 0 to T and have no frequency components above f_U. Then we can represent ℓ_i as

$$\ell_i(t) = \sum_{n=-f_U T}^{f_U T} x_i(n)\, e^{j2\pi nt/T} .$$

Then all of the available information is contained in a set of vectors

$$X(n) = \begin{bmatrix} x_1(n) \\ x_2(n) \\ \cdots \\ \cdots \\ x_K(n) \end{bmatrix} .$$

Assumption 1a: The components of the ℓ_i's at different frequencies are statistically independent.

This means that the probability of a particular X(n) is not affected by any X(m), $n \neq \pm m$.

Let $s_i(t)$ denote a hypothetical signal component of the output from the i^{th} hydrophone. We shall decompose s_i as

$$s_i(t) = \sum_{n=-f_U T}^{f_U T} y_i(n)\, e^{j2\pi nt/T} .$$

and define $Y(n) = \begin{bmatrix} y_1(n) \\ y_2(n) \\ \cdots \\ \cdots \\ y_K(n) \end{bmatrix}$. The $Y(n)$'s will be assumed statistically

independent also.

We must now choose between two hypotheses:

A) The vectors $X(n)$, $n = -f_U T,\ldots,f_U T$ were produced by the random noise field.

B) The vectors $X(n)$, $n = -f_U T,\ldots,f_U T$ were produced by a combination of signal field and noise field.

Let $F_N(X,n)$ denote the probability density function of X when only noise is present, and $F_S(X,n)$ the probability density function of X when signal plus noise is present. Since $X(n) = X^*(-n)$, we shall use $n = 1,2,\ldots,f_U T$ for our testing.

It can be shown from game theory that the best criterion to use on X to decide whether a signal is present is the Neyman-Pearson likelihood ratio,[5]

$$LR = \left(\prod_{n=1}^{f_U T} F_S(X,n) \right) / \left(\prod_{n=1}^{f_U T} F_N(X,n) \right) .$$

Assumption 2a: The $X(n)$'s are sampled from a random process which is stationary and ergodic.

Assumption 3a: The $x_i(n)$'s indicate an equal distribution of power between their real and imaginary components, i.e.,

$$\langle \{Re(x_i(n))\}^2 \rangle = \langle \{Im(x_i(n))\}^2 \rangle ,$$

where $\langle\ \rangle$ denotes an ensemble average, whether the ensemble is over signal plus noise or over noise only.

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[5] W.B. Davenport and W.L. Root, An Introduction to the Theory of Random Signals and Noise, (McGraw-Hill Book Company, Inc., New York, 1958) Chap. 14.

Assumption 4a: The $X(n)$'s have Gaussian distributions in either the signal plus noise case or in the noise only case. In other words, both noise and signal fields are assumed to be Gaussian noise sources.

Under the ergodic assumption the matrix Q in the third section is related to this section by

$$Q^T(n) = Q^*(n) = \{\langle X(n)X^{*T}(n)\rangle_N\}\ (1/N) \quad .$$

So

$$F_N(X,n) = \text{const}\ e^{-(1/N)\ X^{*T}(n)Q^{*-1}(n)\ X(n)} \quad .$$

Under the signal assumptions in the third section we can form the correlation matrix for the signal plus noise situation by using the $X(n)$'s from the noise field and adding vectors $Y(n)$.

$$\langle\{X(n)+Y(n)\}\{X(n)+Y(n)\}^{*T}\rangle_N = \langle X(n)X^{*T}(n)\rangle_N + \langle Y(n)Y^{*T}(n)\rangle_N$$
$$= N(n)\ Q^*(n) + S(n)\ P^*(n)$$

Thus

$$LR = \left\{\text{const} \prod_{n=1}^{f_U T} e^{-X^{*T}(n)\{N(n)Q^*(n)+S(n)P^*(n)\}^{-1}\ X(n)}\right\} \Big/ \left\{\text{const} \prod_{n=1}^{f_U T} e^{-X^{*T}(n)\{N(n)Q^*(n)\}^{-1}\ X(n)}\right\}$$

$$= \left\{\text{const} \prod_{n=1}^{f_U T} e^{-X^{*T}(n)\{N(n)Q^*(n)+S(n)V^*(n)V^T(n)\}^{-1}\ X(n)}\right\} \Big/ \left\{\prod_{n=1}^{f_U T} e^{-X^{*T}(n)\{N(n)Q^*(n)\}^{-1}\ X(n)}\right\}$$

Recalling from matrix algebra that

$$\{NQ^*+SV^*V^T\}^{-1} = (1/N)Q^{*-1} - \{(S/N^2)(Q^{*-1}V^*)(V^TQ^{*-1})\}/[1+(S/N)V^TQ^{*-1}V^*]$$

we can simplify the expression for LR. (This equation is a special case of $(A+UV^T)^{-1} = A^{-1} - \{(A^{-1}U)(V^TA^{-1})/(1+V^TA^{-1}U)\}$, where A is a square matrix, U and V are column vectors, and A and $(A+UV^T)$ are

assumed nonsingular. This equation is readily verified by

$$(A+UV^T)\{A^{-1}-(1/1+V^TA^{-1}U)\,(A^{-1}U)(V^TA^{-1})\}$$

$$= I - (1/1+V^TA^{-1}U)\cdot UV^TA^{-1} + UV^TA^{-1} \quad -(1/1+V^TA^{-1}U)\{U(V^TA^{-1}U)\,V^TA^{-1}\}$$

$$= I + UV^TA^{-1} - (1/1+V^TA^{-1}U)\{UV^TA^{-1} + (V^TA^{-1}U)(UV^TA^{-1})\}$$

$$= I + UV^TA^{-1} - (1/1+V^TA^{-1}U)\{UV^TA^{-1}\,(1+V^TA^{-1}U)\}$$

$= I$) Using this expression and assuming Q^* and $NQ^*+SV^*V^T$ to be nonsingular,

$$LR = \left\{ \text{const} \prod_{n=1}^{f_U T} e^{-X^{*T}(n)\left\{(1/N(n))Q^{*-1}(n) - \frac{(S(n)/N^2(n))Q^{*-1}(n)V^*(n)V^T(n)Q^{*-1}(n)}{1+V^T(n)\,Q^{*-1}(n)\,V^*(n)(S(n)/N(n))}\right\}X(n)} \right\} \Big/ \left\{ \prod_{n=1}^{f_U T} e^{-X^{*T}(n)\,Q^{*-1}(n)\,X(n)} \right\}$$

$$= \text{const} \prod_{n=1}^{f_U T} e^{(S(n)/N^2(n))\cdot\frac{X^{*T}(n)Q^{*-1}(n)V^*(n)V^T(n)Q^{*-1}(n)X(n)}{1+(S(n)/N(n))V^T(n)Q^{*-1}(n)V^*(n)}}$$

or

$$LR = C_{LR}\, e^{\sum_{n=1}^{f_U T} w(n)},$$

where $w(n) = \dfrac{S(n)}{N^2(n)} \dfrac{X^{*T}(n)Q^{*-1}(n)V^*(n)V^T(n)Q^{*-1}(n)X(n)}{1+(S(n)/N(n))\,V^T(n)Q^{*-1}(n)\,V^*(n)}$.

To see how a filter set can develop $W(n)$, we can re-write this as

$$w(n) = \frac{S(n)}{N^2(n)} \frac{X^{*T}(n)\,(Q^{-1}(n)V(n))^*\,(V^T(n)Q^{*-1}(n))X(n)}{1+(S(n)/N(n))V^T(n)Q^{*-1}(n)V^*(n)}$$

$$= \frac{S(n)}{N^2(n)} \frac{(Q^{*-1}(n)V(n))^{*T}\,X^*(n)\,X^T(n)(Q^{*T-1}(n)V(n))}{1+(S(n)/N(n))\,V^T(n)Q^{*-1}(n)V^*(n)}$$

$$= \frac{S(n)}{N^2(n)} \frac{(Q^{-1}(n)V(n))^{*T}X^*(n)\,X^T(n)\,(Q^{-1}(n)V(n))}{1+(S(n)/N(n))\,V^{*T}(n)Q^{-1}(n)\,V(n)}$$

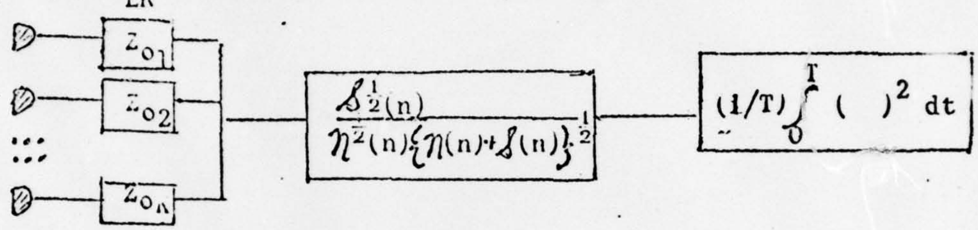
Recall from the third section that $C_0(n) = V^{*T}(n)Q^{-1}(n)V(n)$, and $Z_0(n) = Q^{-1}(n)\,V(n)$, so

$$w(n) = (S(n)/N^2(n))\,(1/1+(S(n)/N(n))C_0(n)\,)\,Z_0^{*T}(n)X^*(n)X^T(n)Z_0(n)$$

This can also be written as

$$W(n) = \left(\mathcal{S}(n)/\eta(n)\{\eta(n)+\mathcal{S}(n)\} \right) Z_0^{*T}(n)X^*(n)X^T(n)Z_0(n) \quad .$$

A detection is said to occur when the likelihood ratio exceeds a certain value, β. This is equivalent to saying that $\sum_{n=1}^{f_U T} W(n)$ exceeds the log of β/C_{Lk}. The desired filter system is



5. OBSERVATIONS

The reader will note the similarity between the transfer function $\mathcal{S}^{\frac{1}{2}}(n)/\eta^{\frac{1}{2}}(n)\{\eta(n)+\mathcal{S}(n)\}^{\frac{1}{2}}$ and the Eckart filter $\mathcal{S}^{\frac{1}{2}}(n)/\eta(n)$. This development, however, did not require the use of a small signal approximation. The filters Z_0 remain the same as those developed in the third section. Thus we have shown that at one frequency the problem of evaluating the Neyman-Pearson likelihood ratio is equivalent to the problem of maximizing the signal-to-noise ratio at the output of a set of linear filters.

A word of caution is in order concerning the assumption of an infinite integration time. This assumption is implicit in the assumption that the frequency components are statistically independent. Thus this development must be used with caution unless long integration times are used.

6. N. WIENER'S APPROACH

The approach of Wiener is to minimize the distortion of the signal. We shall consider a single frequency first. More precisely, if the filter weights are Z_i, and the output is $\sum_{i=1}^{K} Z_i \rho_i(t)$, and $s(t)$ is the

signal at some point in space, the distortion or error in the output of the filters is defined as

$$E = \lim_{T\to\infty} (1/2T)\int_{-T}^{T} \{ s(t)-\sum_{i=1}^{K} z_i \ell_i(t)\} \{s(t)-\sum_{i=1}^{K} z_i \ell_i(t)\}^* dt$$

$$= \lim_{T\to\infty} (1/2T)\int_{-T}^{T} s(t)s^*(t)dt \;-\sum_{i=1}^{K} z_i^* \lim_{T\to\infty} (1/2T)\int_{-T}^{T} s(t)\ell_i^*(t)dt$$

$$-\sum_{i=1}^{K} z_i \lim_{T\to\infty} (1/2T)\int_{-T}^{T} s^*(t)\ell_i(t)\, dt$$

$$+ \sum_{i=1}^{K}\sum_{j=1}^{K} z_i^* z_j \lim_{T\to\infty} (1/2T)\int_{-T}^{T} \ell_i^*(t)\ell_j(t)\, dt \;.$$

The assumption that signal and noise are uncorrelated implies that

$$\lim_{T\to\infty} (1/2T)\int_{-T}^{T} s(t)\, \ell_i^*(t)dt = Se^{j2\pi f \tau_i} \;.$$

Since signal is now present along with the noise,

$$\lim_{T\to\infty} (1/2T)\int_{-T}^{T} \ell_i^*(t)\ell_j(t)\; dt \;= \Theta_{ij} + \rho_{ij} \;.$$

In the previous notation, then, we can write

$$E = \lim_{T\to\infty} (1/2T)\int_{-T}^{T} s(t)s^*(t)dt \;-SZ^{*T}V-SV^{*T}Z+Z^{*T}(SP+NQ)Z \;.$$

In Wiener's formulation, the optimum Z's are those which give the minimum E. To find them

$$dE = -S(dZ)^{*T}V - SV^{*T}(dZ)+(dZ)^{*T}\,(SP+NQ)\,Z+Z^{*T}(SP+NQ)(dZ) \equiv 0 \;\forall (dZ)$$

$$(dZ)^{*T}\{(SP+NQ)Z_W - SV\} +\{Z_W^{*T}(SP+NQ) - SV^{*T}\}(dZ) = 0 \;,$$

or $\quad \mathrm{Re}\{ (dZ)^{*T}[(SP+NQ)Z_W - SV]\} = 0.$

The only way that this can be true for all possible dZ is

$$(SP+NQ)Z_W \;=\; SV$$

$$Z_W \;=\; (SP+NQ)^{-1}SV \;.$$

Recall that $P = VV^{*T}$, so

$$Z_W = (NQ+SVV^{*T})^{-1}\, SV$$

$$= \{(1/N)Q^{-1} -[(S/N^2)(1/1+(S/N)V^{*T}Q^{-1}V)]Q^{-1}V\; V^{*T}Q^{-1}\}\, SV$$

$$\text{or}\quad Z_W = \{(1/N)R - (S/N^2)(1/1+(S/N)\,V^{*T}RV)\ \ RV\,V^{*T}R\}\,SV$$

$$= RV\left\{\frac{S}{N} - \left(\frac{S^2}{N^2}\right)\left(\frac{1}{1+(S/N)V^{*T}RV}\right) V^{*T}RV\right\}.$$

Recall that $G_o = V^{*T}RV$, so

$$Z_W = RV\left\{\frac{S}{N} - \left(\frac{S^2}{N^2}\right)\left(\frac{G_o}{1+G_o\,(S/N)}\right)\right\}$$

Since Z_W is a multiple of Z_o, the array gain of the Wiener filters is G_o.

$$Z_W = Z_o\left\{\frac{S}{N} - \left(\frac{S^2G_o}{N(N+SG_o)}\right)\right\}$$

$$= Z_o\left\{\frac{SN+S^2G_o - S^2G_o}{N(N+SG_o)}\right\}$$

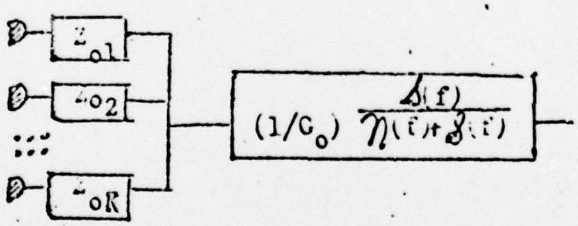
$$= Z_o\,(S/N+SG_o)$$

$$= Z_o(S/N)(1/1+(S/N)G_o\,)$$

$$= Z_o\,(1/G_o)\,(\mathcal{S}/\mathcal{N})(1/1+\mathcal{S}/\mathcal{N}\,)$$

$$= Z_o\,(1/G_o)\,(\mathcal{S}/\mathcal{S}+\mathcal{N})\ .$$

The system indicated by this development would be



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APPENDIX 1. GAUSSIAN DISTRIBUTION OF COMPLEX VARIABLES

The purpose here is to develop the form which a multivariable Gaussian distribution of complex variables must take. To do this, consider a complex random variable $x = a + ib$, where a and b are real numbers. We shall assume that a and b are Gaussian random variables with zero mean, i.e., if $\langle\,\rangle$ denotes the ensemble average, $\langle a\rangle = \langle b\rangle = 0$. The probability distribution functions for a and b, respectively, are

$$P(a) = (1/\sigma_a\sqrt{2\pi})\, e^{-a^2/2\sigma_a^2} \quad \text{and} \quad P(b) = (1/\sigma_b\sqrt{2\pi})\, e^{-b^2/2\sigma_b^2},$$

where $\sigma_a^2 = \langle a^2\rangle$, and $\sigma_b^2 = \langle b^2\rangle$.

We shall make two more assumptions about a and b:

1. a and b are statistically uncorrelated, i.e. $\langle ab\rangle = 0$.
2. a and b have the same variance, i.e., $\langle a^2\rangle = \langle b^2\rangle$.

We shall define

$$\sigma_x^2 = \langle x^* x\rangle = \langle a^2 + b^2\rangle = \langle a^2\rangle + \langle b^2\rangle = \sigma_a^2 + \sigma_b^2 .$$

Under these asumptions we can write $P(x)$ as

$$P(x) = P(a)P(b) = (1/\sigma_x^2\pi)\, e^{-(a^2+b^2)/\sigma_x^2} = (1/\pi\sigma_x^2)\, e^{-x^*x/\sigma_x^2} .$$

Now let $x_1, x_2, \ldots, x_K$ denote K independent complex variables, i.e., $\langle x_i x_j^*\rangle = 0$ if $i \neq j$, and $\langle x_i x_j\rangle = 0\ \forall i, j$. Let $X = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_K \end{bmatrix}$. Then we can write $P(X)$ as

$$P(X) = (1/\pi^K \prod_{i=1}^{K} \sigma_i^2)\, e^{-\sum_{i=1}^{K}(x_i^* x_i/\sigma_i^2)},$$

where $\sigma_i^2 = \langle x_i x_i^*\rangle$.

This can be simplified by defining the $K \times K$ matrices

$$Q_x = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ 0 \cdots & 0 & \cdots & \sigma_K^2 \end{bmatrix}, \text{ and } R_x = Q_x^{-1} = \begin{bmatrix} 1/\sigma_1^2 & 0 & \cdots & 0 \\ 0 & 1/\sigma_2^2 & \cdots & 0 \\ \cdots & & & \\ 0 & 0 & \cdots & 1/\sigma_K^2 \end{bmatrix}.$$

Then we can write $P(X)$ as

$$P(X) = (1/\pi^K |Q_x|)\ e^{-X^{*T} R_x X} \quad .$$

Now consider a second set of random variables, $y_1, y_2, \ldots, y_K$, which are linear combinations of the $x_1, x_2, \ldots, x_K$. We can define $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_K \end{bmatrix}$,

and write $Y = AX$, where A is a nonsingular matrix whose dimension is KxK. To relate the ensemble averages $\langle y_i y_j^* \rangle$ to the x's, we shall form the square matrix $\langle YY^{*T} \rangle$. Note that the ensemble averages in question are the elements of $\langle YY^{*T} \rangle$. But $\langle YY^{*T} \rangle = \langle AXX^{*T} A^{*T} \rangle = A \langle XX^{*T} \rangle A^{*T} = A Q_x A^{*T}$. Thus we can define $Q_y = \langle YY^{*T} \rangle = A Q_x A^{*T}$, and Q_y will always be positive definite hermitian.

We are now in a position to write the probability density function of Y, for it is the probability density function of the corresponding X, i.e.,

$$P(Y) = (N/\pi^K |Q_x|)\ e^{-(A^{-1}Y)^{*T} R_x (A^{-1}Y)}$$

$$= (N/\pi^K |Q_x|)\ e^{-Y^{*T} (A^{-1})^{*T} R_x A^{-1} Y} \quad ,$$

where N is a normalizing factor to assure that the integral of $P(Y)$ over all y_i equals 1.

Then we can define $R_Y = (A^{-1})^{*T} R_x A^{-1}$, noting that $R_y = Q_y^{-1}$. We can now write

$$P(Y) = (N/\pi^K |Q_x|)\ e^{-Y^{*T} R_y Y} \quad .$$

Thus we can expect that under assumptions 1 and 2, the Gaussian distribution for complex variables should take the form of a constant times e to a quadratic form in Y, and that the matrix in this quadratic form will be the inverse of the positive definite hermitian matrix $\langle YY^{*T} \rangle$. Note that the argument is reversible since any hermitian matrix can be diagonalized.